Statistical Modeling of Fleet Data

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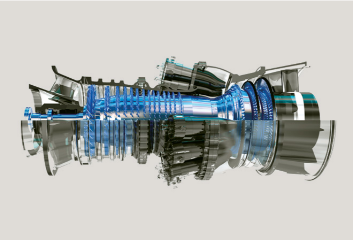
Introduction to *Fleet Data*

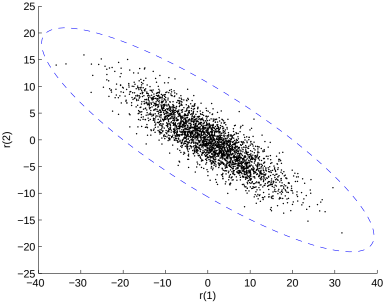
Consider the following problem:

- When GE sells a gas turbine, they offer a *Monitoring and Diagnostics* service
- GE has 1150 turbines currently being monitored, (and hundreds are the 7FA model) (Source: Orbit, Vol. 31)
- Data must be processed remotely (GE M&D, Atlanta)

How do we (tractably) generate a model? How do we use this model to detect anomalies?



Review: Confidence ellipsoid & exceedance monitoring



- Consider the ellipsoid $\{y \in \mathbb{R}^2 | (y - Bx)^T \Sigma^{-1} (y - Bx) \leq \alpha\}$.
- Represents a confidence bound.
- If we know $y \sim \mathcal{N}(Bx, \Sigma)$ in normal operation, then we use Bx and Σ for *anomaly detection*.
- Bx alone doesn't help us, need an accurate Σ.
- We will focus on finding B and Σ (x is assumed to be known).

What is fleet data?

Dataset is hierarchical:

- *Unit:* There exists N "units," so the data is divided into N subsets (indexed $i = 1, ..., N$).
- *Time:* Each of the N subsets contains T data points (indexed $t = 1, ..., T$).
  - t interpreted as sample time.
  - T is the same for each unit (i.e. each unit has the same number of data points).
- *Input/Output:* Each unit has an input output structure that is known *a priori* (input is $x_i(t)$, output is $y_i(t)$).
- *Multivariate data points* Each output data point $y_i(t)$ and input data point $x_i(t)$ is a vector in $\mathbb{R}^{n_y}$ and $\mathbb{R}^{n_x}$, respectively.

Therefore, we can summarize the data structure compactly by writing

$$\left\{ \{x_i(t), y_i(t)\}_{t=1}^{T} \right\}_{i=1}^{N}$$

Objectives

We propose a regression model of the following form:

$$y_i(t) = B_i x_i(t) + v_i(t)$$

with the following known variables:

- $y_i(t) \in \mathbb{R}^{n_y}$ is the (known) output of unit i at time t.
- $x_i(t) \in \mathbb{R}^{n_x}$ is the (known) exogenous input for unit i at time t.

We want to choose B_i such that:

- each $v_i(t)$ has low covariance.
- B_i is "similar" to B_j, $\forall\ i \neq j$
- The distribution of $v_i(t)$ is "similar" to that of $v_j(t)$.

We want reasonable computational complexity.

Regression approach

Recall the assumption:

$$y_i(t) = B_i x_i(t) + v_i(t)$$

For this specific approach, further define:

- The residual:
  - $v_i(t)|S \sim \mathcal{N}(0, S)$, i.i.d.
  - $S \in \mathbb{S}_+^{n_y}$ is the (unknown) residual covariance.
- The unit linear model:
  - $B_i|B_{\text{true}} \sim N_{n,k}(B_{\text{true}}, S/\alpha, I)$, i.i.d.
  - $N(\cdot,\cdot,\cdot)$ denotes the *matrix normal distribution*. For us, for each column of B_i, $b_i \sim \mathcal{N}(b_{\text{true}}, S/\alpha)$.
  - α is a weight, chosen *a priori*.
- No prior information is given about B_{true} or S.

MAP estimation of parameters

We find of $B_1, \dots, B_N$, B_{true}, and S which minimize the negative log likelihood function.
Using the law of total probability,

$$\ell(B_1, \dots, B_N, B_{\text{true}}, S|\mathcal{X}, \mathcal{Y}) = -\log f(\mathcal{X}, \mathcal{Y}|B_1, \dots, B_N, B_{\text{true}}, S)$$

$$= -\log \left(\prod_{i=1}^{N} \prod_{t=1}^{T} f(v_i(t)|B_1, \dots, B_N, B_{\text{true}}, S) \right)$$

$$= \sum_{i=1}^{N} \left(-\log f(B_i|B_{\text{true}}, S) - \sum_{t=1}^{T} \log f(v_i(t)|B_i, S) \right)$$

We have omitted the terms $f(B_{\text{true}})$ and $f(S)$, as there is no prior information about B_{true} or S (we can consider them to have improper priors).

MAP estimation of parameters

Plugging in the log-likelihoods and simplifying:

$$\ell = NT \log |S| + \sum_{i=1}^{N} \Bigg(\alpha \operatorname{tr}\left((B_i - B_{\text{true}})^T S^{-1} (B_i - B_{\text{true}}) \right) \\ + \operatorname{tr}\left((Y_i - B_i X_i)^T S^{-1} (Y_i - B_i X_i) \right) \Bigg)$$

- X_i and Y_i are data matrices
- constant terms are omitted
- Convex in the variables $S^{-1}B_i,\ S^{-1}B_{\text{true}},\ S^{-1}$

Normal equations

The normal equations are found by differentiating w.r.t. the (matrix) variables. They are:

- Unit Model:

$$Y_i X_i^T = \alpha(\widehat{B}_i - B_{\text{true}}) + \widehat{B}_i X_i X_i^T$$

- Average Model:

$$\widehat{B}_{\text{true}} = (1/N) \sum_{i=1}^{N} B_i$$

- Fleetwide Residual Covariance:

$$\widehat{S} = \frac{1}{NT} \sum_{i=1}^{N} \Big(\alpha(B_i - B_{\text{true}})(B_i - B_{\text{true}})^T + (Y_i - B_i X_i)(Y_i - B_i X_i)^T \Big)$$

Are we satisfied?

We chose B_i, B_{true} and S to minimize:

$$\ell = NT \log|S| + \sum_{i=1}^{N} \Bigg(\alpha \operatorname{tr}\left((B_i - B_{\text{true}})^T S^{-1} (B_i - B_{\text{true}}) \right) + \operatorname{tr}\left((Y_i - B_i X_i)^T S^{-1} (Y_i - B_i X_i) \right) \Bigg)$$

We wanted to encode the idea:

- each $v_i(t)$ should have low covariance. (Satisfied)
- B_i should be "similar" to B_j, $\forall\ i \neq j$ (Satisfied)
- The covariance of $v_i(t)$ should be "similar" to that of $v_j(t)$. (Unsatisfied)

Covariance approach

Recall the assumption:

$$y_i(t) = B_i x_i(t) + v_i(t)$$

For this specific approach, further define:

- The residual:
  - $v_i(t)|S \sim \mathcal{N}(0, S_i)$, i.i.d.
  - $S_i \in \mathbb{S}_+^{n_y}$ is the (unknown) residual covariance.
- The unit covariance:
  - $S_i \in \mathbb{S}_+^n$, where $S_i|S_{\text{true}} \sim \mathcal{W}(S_{\text{true}}/p, p)$, i.i.d.
  - $\mathcal{W}(\cdot,\cdot)$ is the Wishart distribution (for random matrix $Z \in \mathbb{R}^{n\times\nu}$, with columns of Z normally distributed, zero-mean i.i.d. random vectors with covariance Σ, then $ZZ^T \sim \mathcal{W}(\Sigma, \nu)$).
  - $p \in \mathbb{R}$ is a (known) weight parameter
- We have no prior information about B_i or S_{true}.

MAP estimation of parameters

We find the MAP estimates of S_{true}, and B_i and S_i, for all $i = 1, ..., N$.
Maximize the log likelihood function:

$$
\begin{aligned}
&\ell(S_1, ..., S_N, B_1, ..., B_N, S_{\text{true}} | \mathcal{X}, \mathcal{Y}) \\
&\quad = -\log \left(\prod_{i=1}^{N} \prod_{t=1}^{T} f(v_i(t) | S_1, ..., S_N, B_1, ..., B_N, S_{\text{true}}) \right) \\
&\quad = \sum_{i=1}^{N} \left(-\log f(S_i | S_{\text{true}}) - \sum_{t=1}^{T} \log f(v_i(t) | S_i, B_i) \right)
\end{aligned}
$$

As before, we ignore the terms $f(S_{\text{true}})$ and $f(B_i)$.

MAP estimation of parameters

Plugging in the log-likelihoods and simplifying:

$$= Np \log |S_{\text{true}}| + \sum_{i=1}^{N} \Bigg(p \operatorname{tr}\left(S_{\text{true}}^{-1} S_i\right) + (T + n + 1 - p) \log |S_i|$$

$$+ \operatorname{tr}\Big((Y_i - B_i X_i)^T S_i^{-1} (Y_i - B_i X_i)\Big)\Bigg)$$

With change of variables, $S_i^{-1} = P_i$, $L^T L = S_{\text{true}}^{-1}$, and $\widetilde{B}_i = S_i^{-1} B_i$, this is convex for large T:

$$= -Np \log \left| L^T L \right| + \sum_{i=1}^{N} \Bigg(p \operatorname{tr}\left(L^T P_i^{-1} L\right) - (T + n + 1 - p) \log |P_i|$$

$$+ \operatorname{tr}(Y_i^T P_i Y_i) - 2 \operatorname{tr}(Y_i^T \widetilde{B}_i X_i) + \operatorname{tr}((\widetilde{B}_i X_i)^T P_i^{-1} (\widetilde{B}_i X_i))\Bigg)$$

Normal equations

Two of the normal equations are found by differentiating w.r.t. the new variables, then substituting back into the natural variables:

- Unit Model:

$$Y_i X_i^T = \widehat{B}_i X_i X_i^T$$

- Average Covariance:

$$\widehat{S}_{\text{true}} = 1/N \sum_{i=1}^{N} S_i$$

- Unit Covariance:

$$0 = \widehat{S}_i(-pS_{\text{true}}^{-1})\widehat{S}_i - (1 + n + T - p)\widehat{S}_i + Y_i Y_i^T - B_i X_i X_i^T B_i^T$$

Normal equations

$$0 = \widehat{S}_i(-pS_{\text{true}}^{-1})\widehat{S}_i - (1 + n + T - p)\widehat{S}_i + Y_iY_i^T - B_iX_iX_i^TB_i^T$$

If we consider another change of variables:

- $Q^{(r)} = Y_iY_i^T - B_iX_iX_i^TB_i^T$
- $A^{(r)} = -(1/2)(1 + n + T - p)I$
- $B^{(r)} = I$
- $R^{(r)} = (1/p)S_{\text{true}}$

This can be solved easily as an algebraic Riccati equation:

$$A^TX + XA - XBR^{-1}B^TX + Q = 0$$

Normal equations

Summarizing:

$$Y_i X_i^T = \widehat{B}_i X_i X_i^T$$

$$0 = \widehat{S}_i(-p S_{\text{true}}^{-1})\widehat{S}_i - (1 + n + T - p)\widehat{S}_i + Y_i Y_i^T - B_i X_i X_i^T B_i^T$$

$$\widehat{S}_{\text{true}} = 1/N \sum_{i=1}^{N} S_i$$

The following algorithm can be used to obtain S_i, S_{true}, and B_i for all i

1. Compute $\widehat{B}_i$
2. Initialize $\widehat{S}_i$ and $\widehat{S}_{\text{true}}$.
3. Compute $\widehat{S}_i$
4. Compute $\widehat{S}_{\text{true}}$
5. Check the variables have converged. If so, stop. If not, go to step 3.

Convergence is guaranteed, by convexity

Now are we satisfied?

We chose B_i, S_{true} and S_i to minimize:

$$\ell = Np \log |S_{\text{true}}| + \sum_{i=1}^{N} \Big(p \operatorname{tr}\left(S_{\text{true}}^{-1} S_i\right) + (T + n + 1 - p) \log |S_i| + \operatorname{tr}\left((Y_i - B_i X_i)^T S_i^{-1} (Y_i - B_i X_i)\right) \Big)$$

We wanted to encode the idea:

- each $v_i(t)$ should have low covariance (satisfied)
- B_i should be "similar" to B_j, $\forall\ i \neq j$ (unsatisfied)
- The covariance of $v_i(t)$ should be "similar" to that of $v_j(t)$. (satisfied, though not obvious)

Brief simulation example

$$y_i(t) = B_i x_i(t) + v_i(t)$$

Random variables were generated according to:

- $x_i(t) \in \mathbb{R}^n$ $x_i(t) \sim \mathcal{N}(0, \Sigma_x)$, i.i.d. the (known) input of unit i at time t.
- $v_i(t) \in \mathbb{R}^n$, $v_i(t) \sim \mathcal{N}(0, S_i)$, i.i.d., is the residual for unit i at time t.
- $B_i \in \mathbb{R}^{n \times k}$, $B_i \sim N_{n,k}(B_{\text{true}}, S_i, I)$ is the static linear map for turbine i.
- $S_i \sim \mathcal{W}(S_{\text{true}}/p, p)$ is the covariance of the residual, with p degrees of freedom.

Constants for simulation

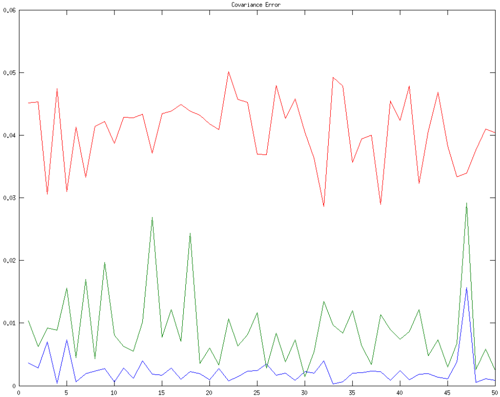
$$\Sigma_x = \begin{bmatrix} 1 & 0.5 & 0.25 & 0 \\ 0.5 & 1 & 0.5 & 0 \\ 0.25 & 0.25 & 1 & 0 \\ 0 & 0 & 0 & 0.001 \end{bmatrix} \qquad S_{\text{true}} = \begin{bmatrix} 0.1 & 0.01 \\ 0.01 & 0.001 \end{bmatrix}$$

$$B_{\text{true}} = \begin{bmatrix} -29.62 & -29.36 & 1 & 0.0733 \\ 0.0314 & 0.0385 & 0.947 & -9.51 \times 10^{-5} \end{bmatrix}$$

$$T = 100 \qquad N = 50 \qquad p = 10 \qquad \alpha = 1$$

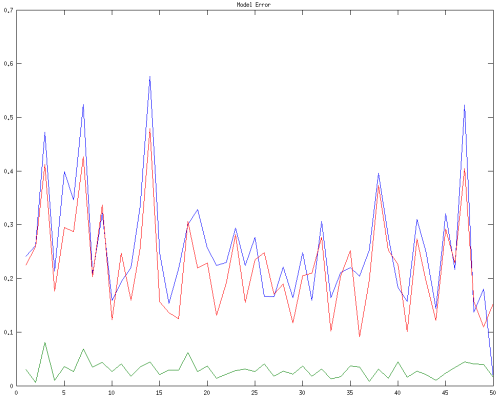
B_{true} obtained from "Performance monitoring of gas turbines," *Journal of Orbit*, Vol. 25, 2005

Results (unit covariance Error)



The error $\|S_i - \widehat{S}_i\|$ vs. i, for the regression model (green), the covariance model (blue), and the naive model (red)

Results (unit model error)



The error $\|B_i - \widehat{B}_i\|$ vs. i, for the regression model (green), the covariance model (blue), and the naive model (red)

Future work

- Using these approaches on real data will prove their efficacy. We are actively seeking such (real) data.
- When are these formulations better than naive approaches? When are they not?
- Is there a formulation that will acheive our original objectives? As a reminder:
  - each $v_i(t)$ should have low covariance
  - B_i should be "similar" to B_j, $\forall\ i \neq j$
  - The covariance of $v_i(t)$ should be "similar" to that of $v_j(t)$.
- Is the Wishart distribution the best prior for the unit covariances?

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