EE292K Project Report: Stochastic Optimal Power Flow in Presence of Renewables

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Abstract

In this work we propose optimal power flow problem maximizing social welfare in the presence of renewables. We use online learning approach to do the stochastic optimization, which also means we do not assume availablity of prior knowledge of the distribution functions involved. We restric our work to power systems with tree topology which allowed us to solve the exact problem (no DC approximation required). We also study optimal locational marginal prices and congestion cost of transmission line in presence of renewables. In addition we formulate OPF to take into account diurnal patterns in renewables.

1 Introduction

In deregulated electricity market, electrical energy is bought and sold like any other comodity. In one of the model we have three key players in the market, generators, load and independent grid operator (IGO). Generators are sellers of electricity, they submit their cost function to IGO, similarly load are buyers and they submit their utility function (or economic benefit function) to IGO. Given cost and utility functions IGO decides quantity of energy bought and sold from/to buyers and sellers. While making these decisions, IGO takes into account the physical constraints associated with power flow and maximum capacity of transmission lines between various generators and loads. More specifically IGO solves Social Welfare maximization problem, the outome also includes locational marginal prices (associated with each bus) and congestion cost for each transmission line [3].

There have been growing support for the clean generation of energy, and adding renewable generators to the grid. Two main characteristics of the renewable energy are: (a) It is quite cheaply produced, (b) it is non-deterministic, unreliable & variable. The renewable energy generation might also be supported by the government incentives, which means cost function could be negative valued (we would not consider this case though). The unreliable and variable nature is more prominent in case of wind and solar energy because weather condition changes and cannot be reliably predicted. It is challenging to develop a good scheme for social welfare. In general, the variability in renewable generation also leads to variablity in power injection and LMP.

When renewables are present it is more meaningful to maximize the average social welfare under average constraints on transmission line capacity. In general, we need to know distribution function of the renewable generator's capacity to solve a stochastic optimization problem. But distribution function may not be available, in this work we are proposing online learning (or sampling) based method to solve such problem. More specifically we are using stochastic subgradient method to solve the optimization problem ([4], [5]). As we will see later, such an approach is not adversly affected by number of renewables and we need not discretise the renewable generation capacity.

Finally we will explore the diurnal pattern [2] of renewables (e.g. wind). It refers to the phenomenon that for specific season wind would start to blow and become calm at specific time of day. Since such a pattern captures most of the variation in the wind, we can use rolling horizon method to maximize social welfare over just one day.

The physical constraints of the power system involves non-linear functions [1] which lead to non-convex optimization problem. Traditionally this has been resolved by taking DC approximation [3]. DC approximation includes three key assumptions, (a) transmission lines are lossless (no resistive component), (b) bus voltage magnitudes are close to unity (in per unit), (c) bus voltage phases are quite close to each other. These assumptions might not hold, for instance, renewables might not have close phase angles due to absence of actual rotating parts. Recent results [6] have shown that if power system network follow specific topology (e.g. tree) then it might be possible to transform the optimal power flow (OPF) problem to a convex problem and solve it exaclty.

2 System Model

We model power system as network with node representing bus and edge representing transmission line. We assume that each bus is either a generator bus or load bus. A generator could be either renewable or non-renewable Figure 1. Note that a bus, with multiple generators or loads connected to it, can be modeled by splitting the bus into multiple busses and connecting them through low impedence transmission line. Let us assume that there are N busses, indexed from 1 to N and J transmission lines, indexed from 1 to J (transmission line can also be indexed by two bus indexes at either end of it). Also, set of all load busses as B^L, set of all renewable generator busses as B^R and set of all non-renewable generator busses as B^{NR}.

Let $P_i \in \mathbb{R}$ be the power injected into the system at bus i. This would typically be positive for generator bus and negative for load bus. Since we want to maximize social welfare, let us define cost and utility functions. If bus i is the load bus let at $P_i = x$, $U_i(x)$ be its utility function. Similarly, if bus i is the generator bus, let $C_i(x)$ be the cost of $P_i = x$ power generation. For simplicity of notation, for any bus i, we define utility and cost function as: $C_i(x) = -U_i(x)$ Then the social welfare is given as $\sum_{i\in B^L} U_i(P_i) - \sum_{i\in B^R, B^{NR}} C_i(P_i) = -\sum_{i=1}^{N} C_i(P_i)$.

Let $V_i = |V_i|e^{j\theta_i}$ be the voltage at bus i, and $\mathbf{V}$ be corresponding vector. Let $\mathbf{Y} = \mathbf{G} + j\mathbf{B}$ ($\mathbf{Y} \in \mathbb{C}^{N\times N}$) be the admittance matrix of the network of transmission lines. Element of $\mathbf{Y}$, $Y_{mn} = G_{mn} + jB_{mn}$, are linearly related to admittance of transmisison line (connecting bus-m and bus-n) $y_{mn} = g_{mn} + jb_{mn}$. Let I_i be the current injected into the network at bus i, and $\mathbf{I}$ is corresponding vector. Bus current and bus voltages are related as, $\mathbf{I} = \mathbf{YV}$.

The bus power injection P_i is constraint by bus voltages as, $P_i = \sum_{k=1}^{N} |V_i||V_k|(G_{ik}cos\theta_{ik} + B_{ik}sin\theta_{ik})$. The difference between power generated and power consumed is the heat dissipated in the transmission lines. The heat loss in transmission line (m,n) as a function of bus voltage is given as $L_{mn}(\mathbf{V}) = |V_m - V_n|^2 g_{mn}$

3 Optimal Power Flow (OPF)

In this section we will go through some schemes of optimal power flow based on social welfare maximization.

3.1 Deterministic OPF

We formulate social welfare maximization optimal power flow problem as follows:

$$\underset{\mathbf{V},\mathbf{P}}{\text{minimize}} \quad \sum_{i=1}^{N} C_i(P_i) \tag{1}$$

$$\text{subject to:} \quad P_i = \sum_{k=1}^{N} |V_i||V_k|(G_{ik}cos\theta_{ik} + B_{ik}sin\theta_{ik}),\ i = 1,\dots,N \tag{2}$$

$$L_i(\mathbf{V}) \leq l^{max},\ i = 1,\dots,J \tag{3}$$

$$V_{min} \leq |V_i| \leq V_{max},\ i = 1,\dots,N \tag{4}$$

$$P_i \leq P_i^{max},\ i \in B^R \tag{5}$$

$$P_i \leq 0,\ i \in B^L \tag{6}$$

Note that objective (1) corresponds to maximizing social utility. Equation (2) represent physical constraint on $\mathbf{P}$ ($\theta_{ik} = \theta_i - \theta_k$), it also implies that supplied power equals power consumed and heat loss. There is physical limit to how much heat can be dissipated in transmission line without any damage, l^{max} in (3) is that limit. This constraint is sometimes [3] approximated as maximum power transmission capacity of line. Equation (4) is bus voltage magnitude limit constraint, which is assumed to be unity in some linear models. P_i^{max} in (5) is maximum power generation capacity of renewable generators, for instance in case of wind mill P_i^{max} depends on wind speed. Finally, (6) means that load cannot inject power into the system.

The optimization problem in this form is hard to solve. Interestingly, a subset of this general problem can be tranformed into a convex problem. Let $\mathbf{W} \in \mathbb{S}_+^N$ be a hermitian symmetric positive semidefinite matrix of rank 1 (W_{mn} being its element), then we do following change of variables (to make the functions linear):

$$\mathbf{W} = \mathbf{V}\mathbf{V}^H \tag{7}$$

$$\sum_{k=1}^{N} |V_i||V_k|(G_{ik}cos\theta_{ik} + B_{ik}sin\theta_{ik}) = \Re(\sum_{k=1}^{N} W_{ik}Y_{ik}^*) \tag{8}$$

$$L_{mn}(\mathbf{W}) = g_{mn}(W_{ii} + W_{kk} - W_{ik} - W_{ki}) \tag{9}$$

Refer to [6] for more details. Now consider following modified optimization problem:

$$
\begin{aligned}
\underset{\mathbf{W},\mathbf{P}}{\text{minimize}} \quad & \sum_{i=1}^{N} C_i(P_i) && (10)\\
\text{subject to:} \quad & \mathbf{P} = \Re(diag(\mathbf{W}\mathbf{Y}^H)) \longleftrightarrow \mu && (11)\\
& L_i(\mathbf{W}) \leq l^{max},\ i = 1, \ldots, J \longleftrightarrow \lambda && (12)\\
& V_{min}^2 \leq W_{ii} \leq V_{max}^2,\ i = 1, \ldots, N && (13)\\
& P_i \leq P_i^{max},\ i \in B^R && (14)\\
& P_i \leq 0,\ i \in B^L && (15)\\
& \mathbf{W} \in \mathbb{S}_+^N && (16)\\
& [Rank(\mathbf{W}) = 1] && (17)
\end{aligned}
$$

Here $\mathbb{S}_+^N$ is set of all hermitian symmetric positive semidefinite matrices and $diag(\mathbf{A})$ is vector formed from diagonal entries of $\mathbf{A}$. This problems is equivalent to the one discussed before (1). If we relax $Rank(\mathbf{W}) = 1$ constraint then this problem becomes convex convex optimization problem. [6] showed that under certain conditions the optimal point satisfies the rank constraint. The set of sufficient conditions are: (a) $C_i(x)$ should be convex and increasing; (b) The network topology should be tree; (c) If two buses are connected by transmission line, both should not have lower bound on P_i; We would restrict our problem/system model to satisfy these conditions. Note that to ensure that (c) is satisfied we did not add constraint $P_i \geq 0$ for generator bus. Thus it is possible to have the optimal point as $P_i^* < 0$ for generator, which has interpretation that if it is economically viable generator bus can consume power (its cost/utility would be zero).

Here λ and μ are the dual variables of the corresponding constraints. The optimal value of daul variable provides useful economic information. For our case, μ_i gives marginal price at bus i (Locational Marginal Price - LMP). λ_i is the congestion cost for the transmission line i, in other words, it is the marginal increase in optimal social utility per unit increase in transmission line capacity (or more precisely, per unit increase in l^{max}).

Without any constraints on transmission line loss (12), the optimal LMP at each bus tend to be close to each other. With the line loss constraint (being active), LMP tend to separate from each bus and with the difference of the order of congestion cost λ_i. Without constraint (14) the LMP is close to marginal cost, i.e. $\mu_i^* \approx |\partial C_i(x)/\partial x|_{x=P_i^*}$. Constraint (14) can be thought of as value of cost function becoming extremely high around P_i^{max}. Thus if constraint (14) is active the LMP could be higher then marginal cost.

3.2 Stochastic OPF

The maximum power generation capacity of renewables vary with time i.e. $P_i^{max}(t)$ is function of time (in case of wind mill, it depends on varying wind conditions). One way to formulate optimal power flow problem is to solve above problem (10) independently for every time instant, since we know the present renewable capacity we can solve the problem exactly. In this case constraints would be satisfied for each time instants, which may not be required for all the constraints. For instance, as long as the average heat dissipation is below certain level instantaeneous heat loss could go very high.

Thus we want to formulate OPF to reflect average objective and constraints. Let the random variable (vector) $\mathbf{X}$ takes the value of $P_i^{max}, i \in B^R$. We formulate the following OPF problem:

$$\begin{aligned}
\underset{\mathbf{W}(\mathbf{X}),\mathbf{P}(\mathbf{X})}{\text{minimize}} \quad & E_{\mathbf{X}}\left(\sum_{i=1}^{N} C_i(P_i(\mathbf{X}))\right) \\
\text{subject to:} \quad & \mathbf{P}(\mathbf{X}) = \Re(diag(\mathbf{W}(\mathbf{X})\mathbf{Y}^H)) \\
& E_{\mathbf{X}}(L_i(\mathbf{W}(\mathbf{X}))) \leq l^{max},\ i = 1, \ldots, J \longleftrightarrow \lambda \\
& V_{min}^2 \leq W_{ii}(\mathbf{X}) \leq V_{max}^2,\ i = 1, \ldots, N \\
& P_i(\mathbf{X}) \leq P_i^{max}(\mathbf{X}),\ i \in B^R \\
& P_i(\mathbf{X}) \leq 0,\ i \in B^L \\
& \mathbf{W}(\mathbf{X}) \in \mathbb{S}_+^N
\end{aligned}$$

Note that solution of this problem, $\mathbf{W}(\mathbf{X}), \mathbf{P}(\mathbf{X})$, is function of $\mathbf{X}$. In theory, it can be solved if we assume $\mathbf{X}$ to be discrete random variable with known distribution function, since the problem remains convex. And since we get to know value of the renewable generation for current time instant ($\mathbf{X}_0$), the optimal power flow values could be found, $\mathbf{W}(\mathbf{X_0}), \mathbf{P}(\mathbf{X_0})$. But the problem scales badly with some system parameter. Let number of discrete levels of P_i^{max} be n_s and number of renewable generator be n_r. Then $\mathbf{X}$ can take $(n_s)^{n_r}$ values and it becomes computationally infeasible to solve the OPF. Also, in many cases we do not have knowledge of distribution function of $\mathbf{X}$.

We thus propose an online learning approach to solve this problem. The iterative algorithm is given as:

$$\underset{\mathbf{W}^t,\mathbf{P}^t}{\text{minimize}} \sum_{i=1}^{N} C_i(P_i^t) + \sum_{i=1}^{J} \lambda_i^t \left(L_i(\mathbf{W}^t) - l^{max}\right) \tag{18}$$

$$\text{subject to: } \mathbf{P}^t = \Re(diag(\mathbf{W}^t\mathbf{Y}^H)) \tag{19}$$

$$V_{min}^2 \leq W_{ii}^t \leq V_{max}^2,\ i = 1, \ldots, N \tag{20}$$

$$P_i^t \leq P_i^{max,\,t},\ i \in B^R \tag{21}$$

$$P_i^t \leq 0,\ i \in B^L \tag{22}$$

$$\mathbf{W}^t \in \mathbb{S}_+^N \tag{23}$$

$$\lambda_i^{t+1} = \lambda_i^t + \frac{\alpha}{t}\left(L_i(\mathbf{W}^t) - l^{max}\right) \tag{24}$$

For every time slot t optimization problem is solved (18) and λ_i is updated for next time slot $t+1$. Over the time the algorithm 'learns' the distribution of $\mathbf{X}$, and λ converges to optimal value. Here we are essentially using stochastic subgradient method ([4], [5]) to solve the problem, details are furnished in Appendix (Section 5.1).

3.3 Optimal Distribution

Different distribution on P_i^{max} would lead to different optimal value of social welfare. We would like to know how the optimal value depends on the distribution, keeping average (of renewable generator capacity) constant. Thus we solve the OPF where we modify maximum power constraint to be on an average.

$$
\begin{aligned}
\underset{\mathbf{W}(\mathbf{X}),\mathbf{P}(\mathbf{X})}{\text{minimize}} \quad & E_{\mathbf{X}}\left(\sum_{i=1}^{N} C_i(P_i(\mathbf{X}))\right) && (25)\\
\text{subject to:} \quad & \mathbf{P}(\mathbf{X}) = \Re(diag(\mathbf{W}(\mathbf{X})\mathbf{Y}^H)) && (26)\\
& E_{\mathbf{X}}(L_i(\mathbf{W}(\mathbf{X}))) \leq l^{max},\ i = 1,\ldots,J \longleftrightarrow \lambda && (27)\\
& V_{min}^2 \leq W_{ii}(\mathbf{X}) \leq V_{max}^2,\ i = 1,\ldots,N && (28)\\
& E_{\mathbf{X}}(P_i(\mathbf{X}) - P_i^{max}(\mathbf{X})) \leq 0,\ i \in B^R && (29)\\
& P_i(\mathbf{X}) \leq 0,\ i \in B^L && (30)\\
& \mathbf{W}(\mathbf{X}) \in \mathbb{S}_+^N && (31)
\end{aligned}
$$

It can be shown that optimal point of this problem are independent of $\mathbf{X}$, that is, $(\mathbf{W}^*(\mathbf{X}), \mathbf{P}^*(\mathbf{X})) = (\mathbf{W}^*, \mathbf{P}^*)$ (Section 5.2). Thus an optimal distribution would be one with zero variance. It also suggests that optimal value may decrease on increasing variance of the distribution. Another viewpoint to look at this is that if sufficient storage is available then there is no effect of this wind (say) variablity, we can achieve the same social welfare as in the case of constant wind.

3.4 Rolling Horizon

The wind speed is not identically distributed for all time throughout the day. For example, wind may start blowing later in morning and calm down around night. Thus wind may exhibit diurnal pattern [2]. Hence it would be relevant to formulate the problem which is coupled over just finite number of time slots spanning a day. Let finite number of time slots of a day be indexed from 1 to T,

$$\underset{\mathbf{W}(t),\mathbf{P}(t)}{\text{minimize}} \quad \sum_{t=1}^{T}\sum_{i=1}^{N} C_i(P_i(t)) \tag{32}$$

$$\text{subject to:} \quad \mathbf{P}(t) = \Re(diag(\mathbf{W}(t)\mathbf{Y}^H)) \; t = 1,\ldots,T \tag{33}$$

$$\frac{1}{T}\sum_{t=1}^{T} L_i(\mathbf{W}(t)) \leq l^{max} + \delta_{Li}, \; i = 1,\ldots,J \tag{34}$$

$$V_{min}^2 \leq W_{ii}(t) \leq V_{max}^2, \; i = 1,\ldots,N \tag{35}$$

$$P_i(t) \leq P_i^{max}(t) + \sum_{\tau=1}^{t-1}(P_i^{max}(\tau) - P_i(\tau)) + \delta_{Pi}, \; i \in B^R, \; t = 1,\ldots,T \tag{36}$$

$$P_i(t) \leq 0, \; i \in B^L, \; t = 1,\ldots,T \tag{37}$$

$$\mathbf{W}(t) \in \mathbb{S}_+^N, \; t = 1,\ldots,T \tag{38}$$

Here the objective is to maximize sum of social welfare across all time slots (32). (34) is constriant on heat dissipation averaged across all time slots (ignore δ_{Li} for now). In this problem we also assumed availability of battery for storage of renewable energy, so that excess power can be used in future time slots. Thus $P_i(t)$ at present cannot exceed (36) capacity at present $P_i^{max}(t)$ plus excess energy stored in past $\sum_{\tau=1}^{t-1}(P_i^{max}(\tau) - P_i(\tau))$.

In the above problem the index $t = 1$ represent present time slot and $t > 1$ represent future, we know the value of $P_i^{max}(t = 1)$ and use estimates for $P_i^{max}(t), \; t = 2,\ldots,T$ (rolling horizon). Similarly, we interpret optimal solution, $\mathbf{P}^*(t = 1)$ being fixed power injection for present, and $\mathbf{P}^*(t > 1)$ just being estimates. As we move ahead in time the number of time slots over which we optimize decreases and eventually toward the end of the day it reduces to single time slot ($T = 1$). Also, δ_{Pi} in (36) is excess power left from past time slot (in a way $t < 1$ is past) of that day and δ_{Li} is associated with past heat loss.

Note that in this case a more optimal approach would be a problem which is coupled over multiple days (rather than just multiple time slot in one day). But, most of the variation in the wind is captured by diurnal pattern within a day so even this approach should be close to optimal.

4 Simulation Results

For simulation we used the power system netwrok shown in Figure 1 , with number of bus $N = 12$, number of transmission lines $J = 11$. Transmission line admittance y_{mn} were uniformly distributed within range $[2:8] - j[18:30]$ per unit. The marginal utility/cost function $(\partial U_i(x)/\partial x)$ for load/generators has been shown in Figure 2. The utility function for the load were chosen to be log function and cost function of the generator were quadratic function. Cost function of renewable were chosen to be significant lower than the cost function of non renewables. We used different distributions for P_i^{max} but the average was kept same, $E(P_i^{max}) = 5$ per unit. The maximum limit on line heat dissipation is taken as, $l^{max} = 0.5$ per unit. The maximum bus voltage constraint is $V_{max} = 1.2$ per unit. Finally all the problems formulated are convex and cvx was used to solve them.

4.1 Deterministic OPF simulation

Deterministic OPF was solved with fixed value of $P_i^{max} = 0.5$ for every renewable. Figure 3 shows the optimal power injection thus obtained, to get some perspective marginal utility/cost is also shown. Note that, renewables accounts for most of the power flowing into the network because of low cost. Also loads with higher utility function tends to get more power from the system. Figure 4 shows locational marginal prices at various busses and congestion cost at transmission lines. Note bus-1 & bus-4 separated with congested transmission line-3 have LMPs with huge difference. Intuitively, bus-1 is connected to renewable generator and bus-4 is connected to load with highest utility function value, causing huge power flow from bus-1 to bus-4 and hence congestion on line-3.

4.2 Stochastic OPF simulation

At each iteration (time) of stochastic OPF P_imax was drawn from exponential distribution with $E(P_i^{max}) = 5$. Wind is known to follow exproximately rayleigh distribution, hence exponential distribution for the maximum power capacity. Figure 5 shows convergence of the lagrange multipliers λ_i thiss with iteration counts. Since we used stochastic subgradient method, it may not converge exactly, but still gets very close to optimal value. Figure 6 shows variation of renewable generation P_1^{max} and LMP μ_4 at load bus. Here we note that at the time of high renewable generation the LMP at bus-4 falls and vice-versa. Since cost of renewable is much less, an increase in generated power leads to more power to flow to the load which leads to lower LMP. Figure 7 shows variation in line loss with time (for couple of lines). Note that instantaeneous line loss can go higher than maximum heat dissipation limit ($l^{max} = 0.5$) but average stays below, this leads to higher average social welfare.

The effect of different distributions of P_i^{max} or more specifically effect of different standard deviation in P_i^{max}. Assuming it to be normally distributed about same mean we plotted optimal value of social welfare for different standard deviations, Figure 8 . In general, social welfare tends to decrease with increase in standard deviation. We also observed that congestion cost tends to increase with increse in standard deviation.

5 Appendix

5.1 Online Learning

Let us reformulate the dual problem as follows:

$$
\begin{aligned}
G(\lambda) = \underset{\mathbf{W}(\mathbf{X}),\mathbf{P}(\mathbf{X})}{\text{minimize}} \, & E_{\mathbf{X}} \left(\sum_{i=1}^{N} C_i(P_i(\mathbf{X})) \right) + \sum_{i=1}^{J} \lambda_i E_{\mathbf{X}} \left(L_i(\mathbf{W}(\mathbf{X})) - l^{max} \right) && (39) \\
\text{subject to: } & \mathbf{P}(\mathbf{X}) = \Re(diag(\mathbf{W}(\mathbf{X})\mathbf{Y}^H)) && (40) \\
& V_{min}^2 \leq W_{ii}(\mathbf{X}) \leq V_{max}^2, \; i = 1, \ldots, N && (41) \\
& P_i(\mathbf{X}) \leq P_i^{max}(\mathbf{X}), \; i \in B^R && (42) \\
& P_i(\mathbf{X}) \leq 0, \; i \in B^L && (43) \\
& \mathbf{W}(\mathbf{X}) \in \mathbb{S}_+^N && (44) \\
\underset{\lambda}{\text{maximize}} \, & G(\lambda) && (45)
\end{aligned}
$$

We decouple the problem over values of $\mathbf{X}$, so that for all $\mathbf{X}$ we solve following :

$$
\begin{aligned}
\underset{\mathbf{W}(\mathbf{X}),\mathbf{P}(\mathbf{X})}{\text{minimize}} \, & \sum_{i=1}^{N} C_i(P_i(\mathbf{X})) + \sum_{i=1}^{J} \lambda_i \left(L_i(\mathbf{W}(\mathbf{X})) - l^{max} \right) && (46) \\
\text{subject to: } & \mathbf{P}(\mathbf{X}) = \Re(diag(\mathbf{W}(\mathbf{X})\mathbf{Y}^H)) && (47) \\
& V_{min}^2 \leq W_{ii}(\mathbf{X}) \leq V_{max}^2, \; i = 1, \ldots, N && (48) \\
& P_i(\mathbf{X}) \leq P_i^{max}(\mathbf{X}), \; i \in B^R && (49) \\
& P_i(\mathbf{X}) \leq 0, \; i \in B^L && (50) \\
& \mathbf{W}(\mathbf{X}) \in \mathbb{S}_+^N && (51)
\end{aligned}
$$

and use subgradient method (iterative nature denoted by index t) to maximize $G(\lambda)$.

$$
\lambda_i^{t+1} = \lambda_i^t + \frac{\alpha}{t} E_{\mathbf{X}} \left(L_i(\mathbf{W}(\mathbf{X})) - l^{max} \right) \tag{52}
$$

we can use stochastic subgradient, that is, we need not take $E_{\mathbf{X}}()$ at each iteration but it will average out itself if $\mathbf{X}$ is drawn randomly:

$$
\lambda_i^{t+1} = \lambda_i^t + \frac{\alpha}{t} \left(L_i(\mathbf{W}(\mathbf{X})) - l^{max} \right) \tag{53}
$$

Finally we assume that the values of random variable over time is ergodic process and hence can replace $\mathbf{X}$ by t.

5.2 Optimal Distribution

Let us assume that we can store energy from renewables, for this case following problem defination would be more appropriate:

$$
\begin{aligned}
\underset{\mathbf{W},\mathbf{P}}{\text{minimize}} \quad & E\left(\sum_{i=1}^{N} C_i(P_i)\right) \\
\text{subject to:} \quad & E(L_k) - l^{max} \leq 0 \\
& E(P_i - P_i^{max}) \leq 0 \\
& \mathbf{P} = \Re\{diag(\mathbf{W}\mathbf{Y}^H)\} \longleftrightarrow \mu \\
& \mathbf{W} \in \mathbb{S}_+^N
\end{aligned} \tag{54}
$$

The $E()$ in above formulation can be replaced by time average:

$$
\begin{aligned}
\underset{\mathbf{W},\mathbf{P}}{\text{minimize}} \quad & \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N} C_i(P_i^t)\right) \\
\text{subject to:} \quad & \frac{1}{T}\sum_{t=1}^{T} L_k^t - l^{max} \leq 0 \\
& \frac{1}{T}\sum_{t=1}^{T}(P_i^t - P_i^{max-t}) \leq 0 \\
& \mathbf{P} = \Re\{diag(\mathbf{W}\mathbf{Y}^H)\} \longleftrightarrow \mu \\
& \mathbf{W} \in \mathbb{S}_+^N
\end{aligned} \tag{55}
$$

Since above problem is convex, $\exists\lambda, \delta$, such that following problem has same solution:

$$
\begin{aligned}
\underset{\mathbf{W},\mathbf{P}}{\text{minimize}} \quad & \frac{1}{T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N} C_i(P_i^t)\right) + \sum_{i=1}^{N}\lambda_i\left(\frac{1}{T}\sum_{t=1}^{T} L_k^t - l^{max}\right) + \sum_{j=1}^{M}\delta_j\left(\frac{1}{T}\sum_{t=1}^{T}(P_i^t - P_i^{max-t})\right) \\
\text{subject to:} & \\
& \mathbf{P} = \Re\{diag(\mathbf{W}\mathbf{Y}^H)\} \longleftrightarrow \mu \\
& \mathbf{W} \in \mathbb{S}_+^N
\end{aligned} \tag{56}
$$

which can be simplified as:

$$
\frac{1}{T}\sum_{t=1}^{T}\left[\left(\underset{\mathbf{W},\mathbf{P}^t}{\text{minimize}} \sum_{i=1}^{N} C_i(P_i^t) + \sum_{j=1}^{M}\lambda_j L_j^t + \sum_{i=1}^{N}\delta_i P_i^t\right) - \sum_{j=1}^{M}\lambda_j l^{max} - \sum_{i=1}^{N}\delta_i P_i^{max-t}\right] \tag{57}
$$

This has two implications, (a) Optimization problem at each time instant can be solved independently from other time instants (b) Optimization problem at each time instants are identical. i.e. solution $\mathbf{P^*}_{t1} = \mathbf{P^*}_{t2} = \mathbf{P^*}$. Thus we can solve following optimization problem with intex t dropped:

$$
\begin{aligned}
\underset{\mathbf{W},\mathbf{P}}{\text{minimize}} \quad & \sum_{i=1}^{N} C_i(P_i) \\
\text{subject to:} \quad & L_k \leq l^{max} \\
& P_i \leq E(P_i^{wind}) \text{ (for renewable)} \\
& P_i \leq 0 \text{ (for load bus)} \\
& \mathbf{P} = \Re\{diag(\mathbf{W}\mathbf{Y}^H)\} \longleftrightarrow \mu \\
& \mathbf{W} \in \mathbb{S}_+^N
\end{aligned} \tag{58}
$$

References

[1] J.J. Grainger and W.D. Stevenson. *Power System Analysis*. McGraw-Hill series in electrical and computer engineering: Power and energy. McGraw-Hill, 1994.

[2] H. Holttinen. Hourly wind power variations in the Nordic countries. *Wind Energy*, 2005.

[3] Minghai Liu. *A framework for congestion management analysis*. PhD thesis, Champaign, IL, USA, 2005. AAI3182319.

[4] N.Z. Shor. *Minimization methods for non-differentiable functions*. Springer series in computational mathematics. Springer-Verlag, 1985.

[5] N.Z. Shor. *Nondifferentiable Optimization and Polynomial Problems*. Nonconvex Optimization and Its Applications. Kluwer, 1998.

[6] Baosen Zhang and D. Tse. Geometry of feasible injection region of power networks. In *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*, pages 1508 –1515, sept. 2011.

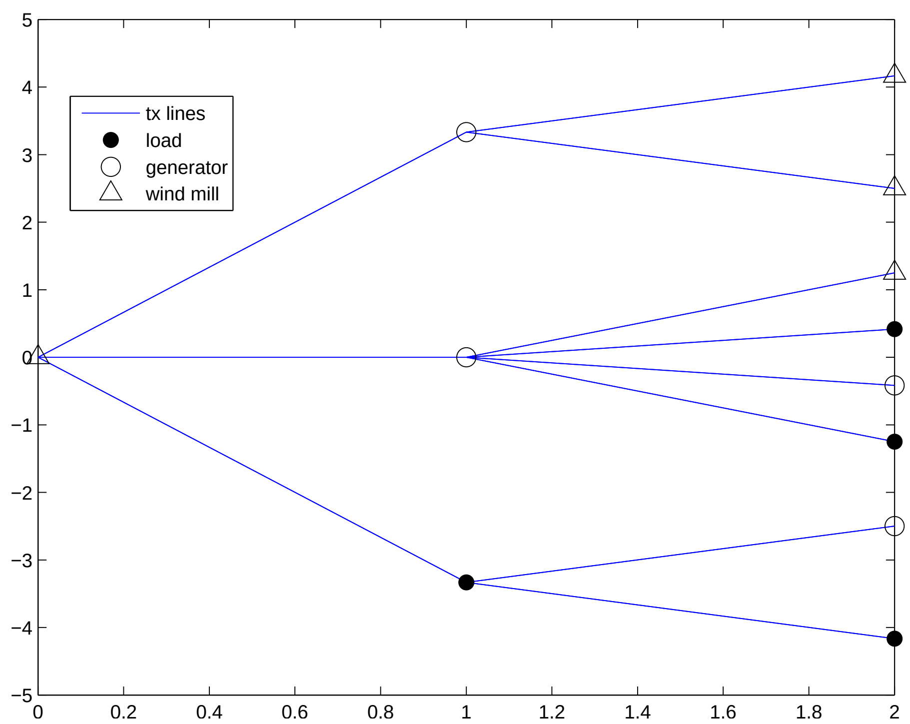

Figure 1: Power system network with $N = 12$ busses (indexed from top to bottom and left to right) and $J = 11$ transmission lines, $B^L = \{4, 8, 10, 12\}$, $B^{NR} = \{2, 3, 9, 11\}$, $B^R = \{1, 5, 6, 7\}$

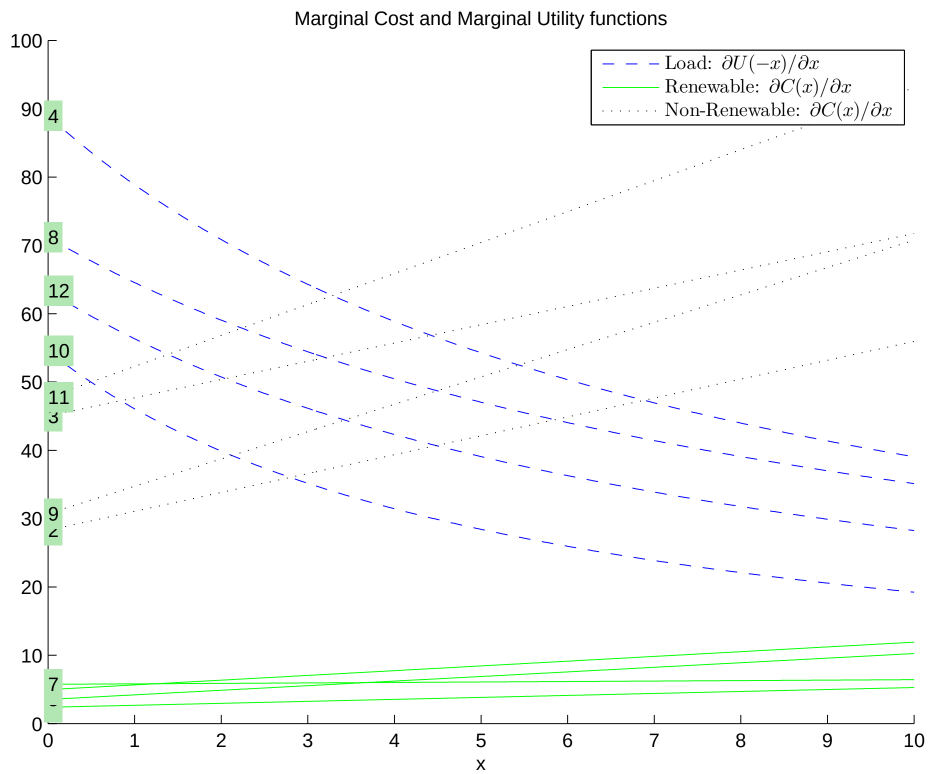

Figure 2: Marginal Cost/Utility functions for generators/loads, labels at start of the curve represent the bus number.

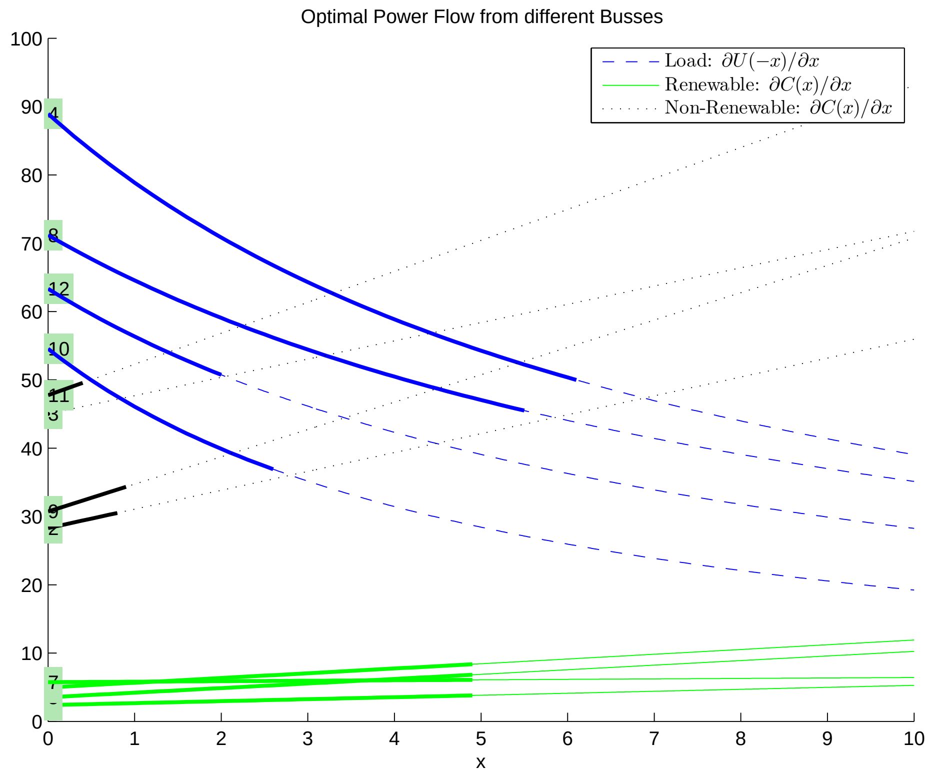

Figure 3: The power injection at various bus has been shown using dark lines on marginal curves.

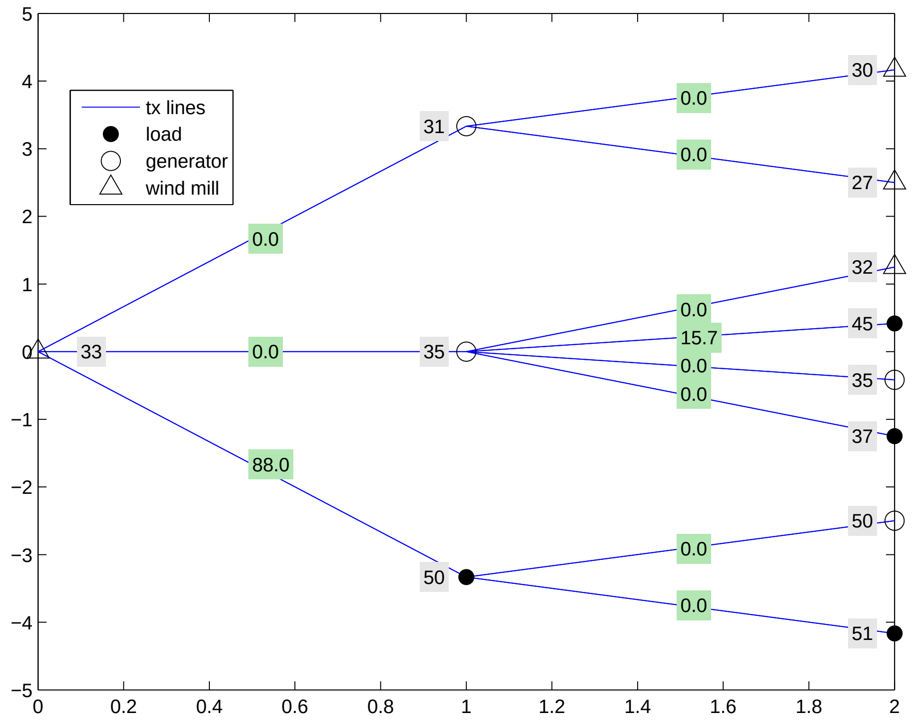

Figure 4: LMP are shown near the bus and congestion cost are shown near center of line

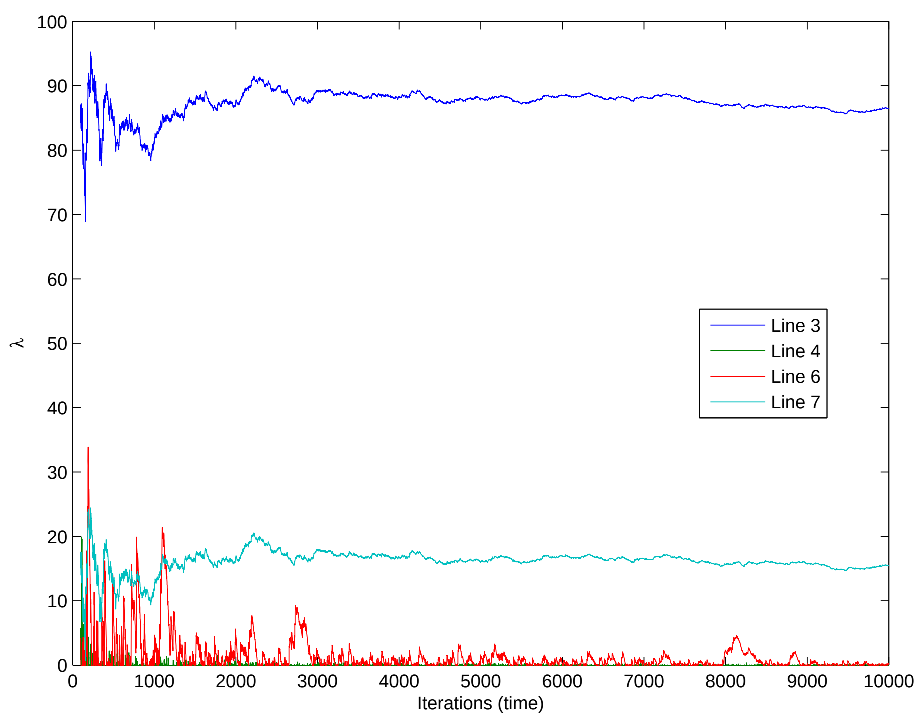

Figure 5: Convergence of lagrange multiplier representing congestion cost of some lines

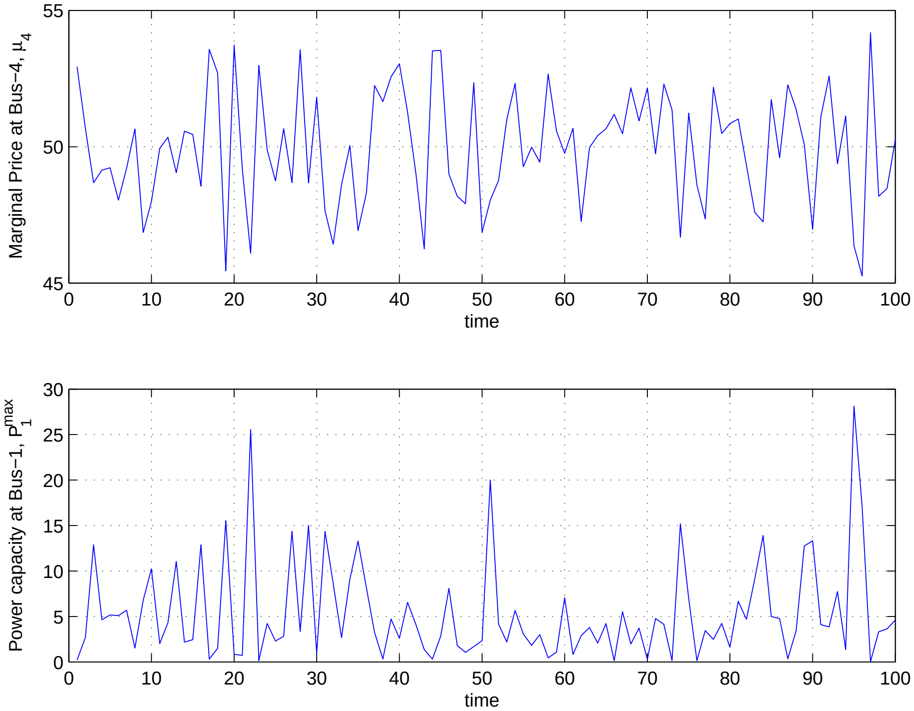

Figure 6: Figure shows the variation of LMP at bus-4 (load bus) with variations of wind (and hence P_1^{max})

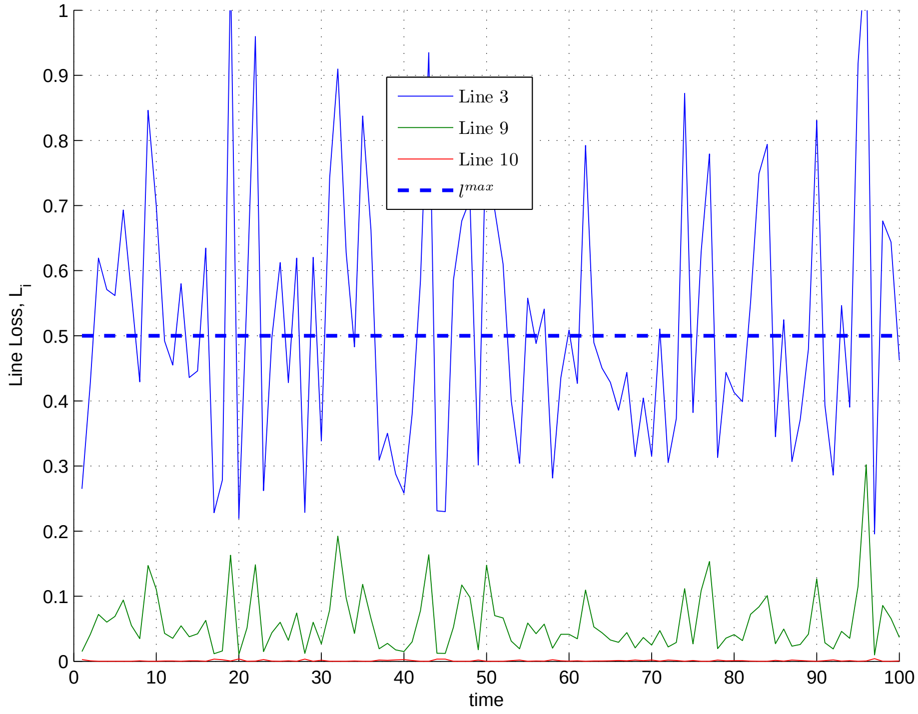

Figure 7: Variation in line loss with time

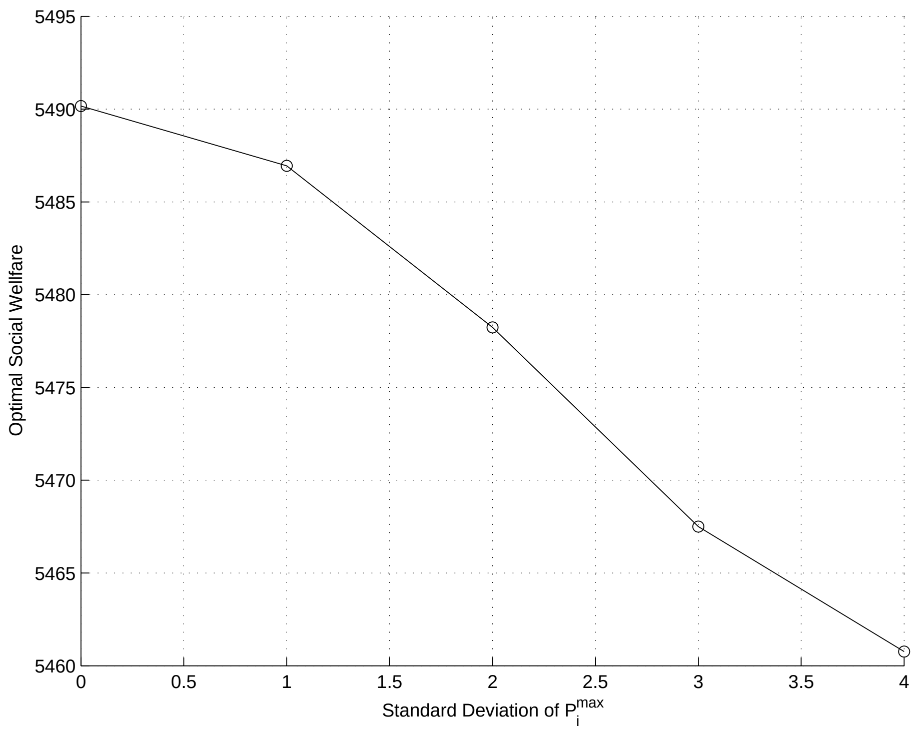

Figure 8: Effect of standard deviation of renewables on social welfare